Introduction to Polymorphic Type Systems

From STLB to parametric and ad-hoc polymorphism

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https://mlaurent.ovh/courses/types/intro.pdf

Why static typing?

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  - What does *wrong* mean here?
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  - Rust: we statically know when memory should be freed (no need for a GC)
- Ecosystem:
  - Type information is useful for the programmer (documentation)
  - Type information is useful for the IDE (refactoring, completion, etc.)

Multiple flavors of types

- More focused on expressivity (not in this lecture):
  - Dependent types (e.g. proof assistants: Lean, Rocq)
    ⇒ Type-checking is not decidable, the programmer must annotate manually
  - Refinement types (e.g. verification-aware languages: F*, Dafny, Why3)
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    but can be partly automated using SMT-solvers

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    ⇒ Type-checking is (usually) not decidable,
    but can be partly automated using SMT-solvers
- More focused on usability:
  - Type systems of mainstream static languages (e.g. Rust, OCaml)
    ⇒ Type-checking is decidable, and we may even have type inference
  - Static type-checkers for dynamic languages (e.g. TypeScript, MyPy, PyRight)
    ⇒ Got more attention in the last 10 years (my research area)

Simply Typed Lambda Calculus (STLC)

Reminder: λ-calculus

Constants	c	::=	$\texttt{true} \mid \texttt{false} \mid 0 \mid 1 \mid 2 \mid 3 \mid \ldots$
Expressions	e	::=	$c \mid x \mid \lambda x.e \mid e\,e \mid (e, e) \mid \pi_1 e \mid \pi_2 e \mid e\,\texttt{?}\,e\,\texttt{:}\,e$
Values	v	::=	$c \mid \lambda x.e \mid (v, v)$

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- **Expressions** e are programs, composed of:
  constants c, variables x, functions $(\lambda x.e)$, function application $(e\ e)$,
  pairs $((e, e))$, pair projections (π_1, π_2), and conditionals $(e\,\texttt{?}\,e\,\texttt{:}\,e)$.

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  constants c, variables x, functions $(\lambda x.e)$, function application $(e\ e)$,
  pairs $((e, e))$, pair projections (π_1, π_2), and conditionals $(e\,\texttt{?}\,e\,\texttt{:}\,e)$.
- **Values** v are results (fully-reduced expressions):
  constants, functions, and pairs of values.
- All values are expressions, but not all expressions are values.

$$
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\textbf{Values} & v & ::= & c \mid \lambda x.e \mid (v,v)
\end{array}
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Call-by-value semantics:

- When we have an application, we first reduce (*evaluate*) the argument, then we enter the function (β-reduction),
- When we have a projection, we first reduce the argument, then we project.

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$(\lambda x.e)v$	$\rightsquigarrow$	$e\{v/x\}$	β-reduction	*(function application)*
$\pi_1(v_1, v_2)$	$\rightsquigarrow$	v_1	Left projection	*(return first element of a pair)*
$\pi_2(v_1, v_2)$	$\rightsquigarrow$	v_2	Right projection	*(return second element of a pair)*
$\texttt{true}\,\texttt{?}\,e_1\,\texttt{:}\,e_2$	$\rightsquigarrow$	e_1	Conditional (1)	*(take first branch)*
$\texttt{false}\,\texttt{?}\,e_1\,\texttt{:}\,e_2$	$\rightsquigarrow$	e_2	Conditional (2)	*(take second branch)*

Example

We assume we have an integer comparison function `leq`:

$$\texttt{leq}\,(n_1, n_2) \rightsquigarrow \begin{cases} \texttt{true} & \text{if } n_1 \leq n_2 \\ \texttt{false} & \text{otherwise} \end{cases}$$

Apply $\texttt{max} \equiv \lambda x.\, \lambda y.\, \texttt{leq}\,(x, y)\,\texttt{?}\,y\,\texttt{:}\,x$ to 42 and 24 and write the reduction steps.

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$$\begin{aligned}
& (\lambda x.\, \lambda y.\, \texttt{leq}\,(x, y)\,\texttt{?}\,y\,\texttt{:}\,x)\ 42\ 24 && (\beta\text{-reduction}) \\
\rightsquigarrow\ & (\lambda y.\, \texttt{leq}\,(42, y)\,\texttt{?}\,y\,\texttt{:}\,42)\ 24 && (\beta\text{-reduction}) \\
\rightsquigarrow\ & \texttt{leq}\,(42, 24)\,\texttt{?}\,24\,\texttt{:}\,42 && (\text{semantics of } \texttt{leq}) \\
\rightsquigarrow\ & \texttt{false}\,\texttt{?}\,24\,\texttt{:}\,42 && (\text{conditional}) \\
\rightsquigarrow\ & 42
\end{aligned}$$

Exercise

Currified function it takes its different parameters (x_1, x_2, ...) successively:
$\lambda x_1.\ \lambda x_2.\ \ldots\ x_1\ \ldots\ x_2\ \ldots$. A currified function can be partially applied.

Uncurrified function it takes its different parameters (x_1, x_2, ...) all at once using a pair/tuple: $\lambda x.\ \ldots\ \pi_1 x\ \ldots\ \pi_2 x\ \ldots$.

Write an uncurrified version of $\texttt{max} \equiv \lambda x.\ \lambda y.\ \texttt{leq}\ (x, y)\ \texttt{?}\ y\ \texttt{:}\ x$,
apply it to 42 and 24 and write the reduction steps.

Simple Monomorphic Types

$$\begin{array}{lrcl} \textbf{Base Types} & b & ::= & \texttt{bool} \mid \texttt{int} \mid \ldots \\ \textbf{Types} & s, t & ::= & b \mid t \to t \mid t \times t \end{array}$$

We have a type constructor for each kind of value of our language:

$$\textbf{Values} \quad v ::= c \mid \lambda x.e \mid (v, v)$$

- Base types (b) represent constants,
- Arrows ($\to$) represent λ-abstractions,
- Products ($\times$) represent pairs.

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Examples:

- Currified `max` has type $\texttt{int} \to \texttt{int} \to \texttt{int}$
  ($\to$ is associative to the right, so $\texttt{int} \to \texttt{int} \to \texttt{int} \equiv \texttt{int} \to (\texttt{int} \to \texttt{int})$)
- Uncurrified `max` has type $(\texttt{int} \times \texttt{int}) \to \texttt{int}$

Typing rules

$$\text{[BoolF]}\ \frac{}{\texttt{false} : \texttt{bool}} \qquad \text{[BoolT]}\ \frac{}{\texttt{true} : \texttt{bool}} \qquad \text{[Int]}\ \frac{}{n : \texttt{int}}$$

$$\text{[Pair]}\ \frac{e_1 : t_1 \qquad e_2 : t_2}{(e_1, e_2) : t_1 \times t_2} \qquad \text{[Cond]}\ \frac{e : \texttt{bool} \qquad e_1 : t \qquad e_2 : t}{e\,\texttt{?}\,e_1\,\texttt{:}\,e_2 : t}$$

$$\text{[App]}\ \frac{e_1 : s \to t \qquad e_2 : s}{e_1\ e_2 : t} \qquad \text{[LProj]}\ \frac{e : t_1 \times t_2}{\pi_1 e : t_1} \qquad \text{[RProj]}\ \frac{e : t_1 \times t_2}{\pi_2 e : t_2}$$

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- Under the line, we have the conclusion,

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- Over the line, we have the premises,
- Under the line, we have the conclusion,
- A rule with no premise ([BoolF], [BoolT], [Int]) is sometimes called an axiom.

What to do for λ-abstractions and variables?

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We need a notion of type environment (or *type context*) to store the type of the variables that are in the current scope.

$$\textbf{Type Environments} \quad \Gamma \ ::= \ \varnothing \mid x : t, \Gamma$$

We can freely reorder bindings in an environment.

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We add environments to our judgments, $\Gamma \vdash e : t$, which reads: under the environment Γ, the expression e has type t.

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Exercise

We consider our built-in comparison `leq` to be a variable of type $(\texttt{int} \times \texttt{int}) \to \texttt{bool}$ in the current scope.

Try to derive the type $\texttt{int} \to (\texttt{int} \to \texttt{int})$ for the function

$$\texttt{max} \quad \equiv \quad \lambda x.\, \lambda y.\, \texttt{leq}\,(x, y)\,?\,y : x$$

under the initial environment $\Gamma \quad = \quad \texttt{leq} : (\texttt{int} \times \texttt{int}) \to \texttt{bool}$.

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$$[\text{App}]\ \frac{\Gamma \vdash e_1 : s \to t \qquad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1\ e_2 : t} \qquad [\text{Abs}]\ \frac{\Gamma, x : s \vdash e : t}{\Gamma \vdash \lambda x.e : s \to t} \qquad [\text{Var}]\ \frac{}{\Gamma, x : t \vdash x : t}$$

For concision, we define:

$$\begin{aligned}
\Gamma &= \texttt{leq} : (\texttt{int} \times \texttt{int}) \to \texttt{bool} \\
\Gamma' &= \texttt{leq} : (\texttt{int} \times \texttt{int}) \to \texttt{bool}, \quad x : \texttt{int} \\
\Gamma'' &= \texttt{leq} : (\texttt{int} \times \texttt{int}) \to \texttt{bool}, \quad x : \texttt{int}, \quad y : \texttt{int}
\end{aligned}$$

Typing derivation for `max`:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots}{\Gamma'' \vdash \texttt{leq}\ (x, y) : \texttt{bool}} [\text{App}] \quad \dfrac{}{\Gamma'' \vdash y : \texttt{int}} \quad \dfrac{}{\Gamma'' \vdash x : \texttt{int}}}{\Gamma'' \vdash \texttt{leq}\ (x, y)\ \texttt{?}\ y : x : \texttt{int}} [\text{Cond}]}{\Gamma' \vdash \lambda y.\ \texttt{leq}\ (x, y)\ \texttt{?}\ y : x : \texttt{int} \to \texttt{int}} [\text{Abs}]}{\Gamma \vdash \lambda x.\ \lambda y.\ \texttt{leq}\ (x, y)\ \texttt{?}\ y : x : \texttt{int} \to (\texttt{int} \to \texttt{int})} [\text{Abs}]$$

Type safety

Theorem (Type safety)

If $\varnothing \vdash e : t$, then either:

- *$e \rightsquigarrow^{\infty}$ (e diverges), or*
- *$e \rightsquigarrow^{*} v$ with $\varnothing \vdash v : t$ (e reduces to a value of the same type)*

In particular, this means that a well-typed expression cannot get stuck (e.g. 42 42).
How to prove this theorem?

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Lemma (Type preservation)

If $\Gamma \vdash e : t$ and $e \rightsquigarrow e'$, then $\Gamma \vdash e' : t$.

Lemma (Progress)

If $\varnothing \vdash e : t$, then either e is a value or $\exists e'.\ e \rightsquigarrow e'$.

Towards an algorithm

$$[\text{App}]\ \frac{\Gamma \vdash e_1 : s \to t \qquad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1\ e_2 : t} \qquad [\text{LProj}]\ \frac{\Gamma \vdash e : t_1 \times t_2}{\Gamma \vdash \pi_1 e : t_1} \qquad [\text{Pair}]\ \frac{\Gamma \vdash e_1 : t_1 \qquad \Gamma \vdash e_2 : t_2}{\Gamma \vdash (e_1, e_2) : t_1 \times t_2}$$

Each rule of our type system is structural: it only applies to one expression constructor.

- [App] only applies on applications,
- [LProj] only applies on left projections,
- [Pair] only applies on pairs, etc.

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- [Pair] only applies on pairs, etc.

This makes our type system syntax-directed: we know which rule to apply just by looking at the syntax of the expression we are typing.

However, the rule [Abs] is not analytic: the domain s of the λ-abstraction does not appear in the initial expression or environment.

$$[\text{Abs}]\ \frac{\Gamma, x : s \vdash e : t}{\Gamma \vdash \lambda x.e : s \to t}$$

While this domain s can be deduced if we know the resulting type (type checking), this is not the case if we want to do type inference (i.e. find the type of the expression).

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We will write the input of the algorithm in green, and its output in blue.

Type checking Finding a derivation for the judgment $\Gamma \vdash e : t$
(i.e. construct a derivation tree by recursively building the premises)

Type inference Finding a type t and a derivation for the judgment $\Gamma \vdash e : t$.

Unification

We will turn our type system into an inference algorithm based on unification.

To that purpose, we add type variables to the syntax of types:

Base Types	b	::=	$\texttt{bool} \mid \texttt{int} \mid \ldots$
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Definition (Most general unifier)

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Example: for two terms $e_1 \equiv \alpha \to \beta$ and $e_2 \equiv \texttt{int} \to \beta$,

- $\{\alpha \rightsquigarrow \texttt{int}\}$ is a most general unifier for e_1 and e_2,
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If there exists a unifier for e_1 and e_2,
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The unification function $\text{mgu}(e_1, e_2)$ can be computed in linear time (cf. `unify` algorithm from your previous lectures).

Exercise

- What is mgu($\texttt{int} \times \beta,\ \ \alpha \times (\alpha \to \texttt{int})$)?
- When there exists a most general unifier, is it always **unique**?
- What is mgu($\alpha,\ \ \beta$)?
- What is mgu($\alpha \to \texttt{int},\ \ \alpha$)?

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To infer the domain of a λ-abstraction, we initially type it with a fresh type variable α, and then substitute it on-the-fly when required, using unification.

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In particular, if we get $\Gamma \vdash e : t \dashv \psi$, then $\Gamma\psi \vdash e : t$ should be derivable.

Axiomatic rules just return the identity substitution, noted $\varnothing$:

$$[\text{BoolF}]\ \frac{}{\Gamma \vdash \texttt{false} : \texttt{bool} \dashv \varnothing} \qquad [\text{BoolT}]\ \frac{}{\Gamma \vdash \texttt{true} : \texttt{bool} \dashv \varnothing} \qquad [\text{Int}]\ \frac{}{\Gamma \vdash n : \texttt{int} \dashv \varnothing}$$

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Note: if no rule can be applied (e.g. we want type x but there is no binding for x in our environment Γ), then the type inference algorithm fails (we get a static type error).

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Note: if no rule can be applied (e.g. we want type x but there is no binding for x in our environment Γ), then the type inference algorithm fails (we get a static type error).

$$[\text{Abs}]\ \frac{\Gamma, x : \alpha \vdash e : t \dashv \psi}{\Gamma \vdash \lambda x.e : (\alpha\psi) \to t \dashv \psi}\ \alpha \text{ fresh}$$

To type $\lambda x.e$, we assume x has a fresh type α and recursively call our algorithm on the body e, yielding a type t and a substitution ψ. The substitution ψ must be applied to our context (in particular to α), so the resulting type for our λ-abstraction is $(\alpha\psi) \to t$.

$$\text{[App]}\ \frac{\Gamma \vdash e_1 : t_1 \dashv \psi_1 \qquad \Gamma\psi_1 \vdash e_2 : t_2 \dashv \psi_2 \qquad \psi = \mathsf{mgu}(t_1\psi_2, t_2 \to \alpha)}{\Gamma \vdash e_1\ e_2 : \alpha\psi \dashv \psi \circ \psi_2 \circ \psi_1}\ \alpha \text{ fresh}$$

To type an application $e_1\ e_2$, we first recursively type e_1, yiedling a type t_1 and a substitution ψ_1. We then type e_2 under the updated context $\Gamma\psi_1$, yielding a type t_2 and a substitution ψ_2.

At this point, the argument has type t_2, and the function has type $t_1\psi_2$. We thus use unification to solve the constraint

$$t_1\psi_2 \equiv t_2 \to \alpha$$

(with α a fresh type variable representing the type of the result),
yielding a substitution ψ. The type of the result of the application is now $\alpha\psi$.

$$[\text{Pair}]\ \frac{\Gamma \vdash e_1 : t_1 \dashv \psi_1 \qquad \Gamma\psi_1 \vdash e_2 : t_2 \dashv \psi_2}{\Gamma \vdash (e_1, e_2) : (t_1\psi_2) \times t_2 \dashv \psi_2 \circ \psi_1}$$

$$[\text{LProj}]\ \frac{\Gamma \vdash e : t \dashv \psi \qquad \psi' = \mathsf{mgu}(t, \alpha_1 \times \alpha_2)}{\Gamma \vdash \pi_1 e : \alpha_1\psi' \dashv \psi' \circ \psi}\ \alpha_1, \alpha_2 \text{ fresh}$$

$$[\text{RProj}]\ \frac{\Gamma \vdash e : t \dashv \psi \qquad \psi' = \mathsf{mgu}(t, \alpha_1 \times \alpha_2)}{\Gamma \vdash \pi_2 e : \alpha_2\psi' \dashv \psi' \circ \psi}\ \alpha_1, \alpha_2 \text{ fresh}$$

$$[\text{Cond}]\ \frac{\begin{array}{c}\Gamma \vdash e : s \dashv \psi \qquad \psi' = \mathsf{mgu}(s, \mathtt{bool}) \\ (\Gamma\psi)\psi' \vdash e_1 : t_1 \dashv \psi_1 \qquad ((\Gamma\psi)\psi')\psi_1 \vdash e_2 : t_2 \dashv \psi_2 \qquad \psi'' = \mathsf{mgu}(t_1\psi_2, t_2)\end{array}}{\Gamma \vdash e\,?\,e_1 : e_2 : t_2\psi'' \dashv \psi'' \circ \psi_2 \circ \psi_1 \circ \psi' \circ \psi}$$

These typing rules are a reformulation of Algorithm W in the context of STLC.

Exercise

Derive a type for $\lambda x.\ (\pi_2 x, \pi_1 x)$ under the empty environment $\varnothing$.

$$\text{[Abs]}\ \frac{\Gamma, x : \alpha \vdash e : t \dashv \psi}{\Gamma \vdash \lambda x.e : (\alpha\psi) \to t \dashv \psi}\ \alpha \text{ fresh}$$

$$\text{[Pair]}\ \frac{\Gamma \vdash e_1 : t_1 \dashv \psi_1 \qquad \Gamma\psi_1 \vdash e_2 : t_2 \dashv \psi_2}{\Gamma \vdash (e_1, e_2) : (t_1\psi_2) \times t_2 \dashv \psi_2 \circ \psi_1}$$

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Lack of polymorphism

What happens if we try to infer the type of the identity function $\lambda x.x$?

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We obtain the type $\alpha \to \alpha$. Let's add it to our context:

$$\Gamma \quad = \quad \texttt{id} : \alpha \to \alpha$$

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Now, what happens if we try to type (`id 42`, `id false`)?

- The first application `id 42` substitutes α by `int`, yielding a new environment where `id` : `int` → `int`,
- Then, the second application `id false` fails.

Parametric polymorphism (Hindley-Milner)

Compositionality

A type system should be compositional: each definition of your program is typed sequentially, only once, without knowing how it will be used by later definitions.

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A type system should be compositional: each definition of your program is typed sequentially, only once, without knowing how it will be used by later definitions.

However, a definition may be use multiple times with arguments of different types:

```
let id x = x (* Type: 'a -> 'a *)
(* ... *)
let foo =
  id 42,  (* 'a should be substituted by int *)
  id true (* 'a should be substituted by bool *)
(* ... *)
let bar x =
  id x    (* 'a should be substituted by the type of x *)
```

$\Rightarrow$ we need a way to use different instantiations of a definition

Let-bindings

In order to model sequential definitions, we add let-bindings to our syntax:

$$\begin{array}{lll} \textbf{Expressions} & e & ::= \quad c \mid x \mid \lambda x.e \mid e\,e \mid (e,e) \mid \pi_1 e \mid \pi_2 e \mid e\,?\,e : e \\ & & \qquad\qquad \mid \texttt{let}\, x = e\, \texttt{in}\, e \\ \textbf{Values} & v & ::= \quad c \mid \lambda x.e \mid (v,v) \end{array}$$

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Semantics: we first reduce the definition, then

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Typing rule (first attempt, no parametric polymorphism):

$$[\text{Let}]\ \frac{\Gamma \vdash e_1 : s \qquad \Gamma, x : s \vdash e_2 : t}{\Gamma \vdash \texttt{let}\, x = e_1\, \texttt{in}\, e_2 : t}$$

Type Schemes

We want a way to give a polymorphic type to a let-definition, that is, a type that can be instantiated in different ways at different locations.

To that purpose, we define a notion of type scheme:

Base Types	b	::=	$\texttt{bool} \mid \texttt{int} \mid \ldots$
Types	s, t	::=	$b \mid t \to t \mid t \times t \mid \alpha$
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where $\vec{\alpha}$ is a set of type variables.

Intuitively, a type scheme $\forall \vec{\alpha}.\, t$ represents a set of types:
it represents all the instances of t obtained by substituting the type variables in $\vec{\alpha}$.

$$\forall \vec{\alpha}.\, t \quad \simeq \quad \{t\psi \mid \psi \in \textbf{Substs}, \mathrm{dom}(\psi) \subseteq \vec{\alpha}\}$$

In particular, if $\vec{\alpha} = \varnothing$, then $\forall \vec{\alpha}.\, t \simeq \{t\}$.

New Type Environments

We update our type environments Γ to associate variables to type-schemes (instead of types):

$$\begin{array}{lrcl} \textbf{Type Schemes} & \sigma & ::= & \forall \vec{\alpha}.\ t \\ \textbf{Type Environments} & \Gamma & ::= & \varnothing \mid x : \sigma, \Gamma \end{array}$$

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For instance, if $(x : \forall \alpha.\ \alpha \to \alpha) \in \Gamma$,
it means that the variable x can be typed with any type in this set:

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Note: our type environments now associate variables to type-schemes,
but our typing rules still derive a type ($\Gamma \vdash e : t$), not a type-scheme.

New Typing rules

As a variable can now be associated to multiple types, we have to modify the [Var] typing rule: it has to choose one type among those captured by the type-scheme.

$$\text{[Var]}\ \frac{}{\Gamma, x : t \vdash x : t} \qquad \longrightarrow \qquad \text{[Var]}\ \frac{}{\Gamma, x : \forall \vec{\alpha}.\ t \vdash x : t\psi}\ \mathrm{dom}(\psi) \subseteq \vec{\alpha}$$

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We also need to modify the typing rules that add a binding in the environment Γ:

$$[\text{Let}]\ \frac{\Gamma \vdash e_1 : s \qquad \Gamma, x : s \vdash e_2 : t}{\Gamma \vdash \texttt{let}\, x = e_1\, \texttt{in}\, e_2 : t} \qquad \longrightarrow \qquad \textbf{???}$$

$$[\text{Abs}]\ \frac{\Gamma, x : s \vdash e : t}{\Gamma \vdash \lambda x.e : s \to t} \qquad \longrightarrow \qquad \textbf{???}$$

Should we allow any type scheme as a domain in [Abs] rules?

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Let's say e_1 is the identity $\lambda y.y$, for which we derive the type $\alpha \to \alpha$.
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Should generalize(s) quantify over all type variables in s?

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No, it is **unsound** to generalize type variables that are bound to the current environment Γ. We should only generalize type variables in $\text{fv}(s) \setminus \text{fv}(\Gamma)$.

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Exercise

Find a derivation for the judgment $\varnothing \vdash \texttt{let}\, x = \lambda y.y\, \texttt{in}\, (x\ 42, x\ \texttt{true}) : \texttt{int} \times \texttt{bool}$.

$$[\text{Int}]\ \frac{}{\Gamma \vdash n : \texttt{int}} \qquad [\text{Bool}]\ \frac{}{\Gamma \vdash b : \texttt{bool}} \qquad [\text{Var}]\ \frac{}{\Gamma, x : \forall \vec{\alpha}.\ t \vdash x : t\psi}\ \mathrm{dom}(\psi) \subseteq \vec{\alpha}$$

$$[\text{Abs}]\ \frac{\Gamma, x : \forall \varnothing.\ s \vdash e : t}{\Gamma \vdash \lambda x.e : s \to t} \qquad [\text{App}]\ \frac{\Gamma \vdash e_1 : s \to t \qquad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1\ e_2 : t}$$

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Type inference

Two rules are not analytic (i.e. we have to guess a type or substitution):

$$[\text{Abs}]\ \frac{\Gamma, x : \forall \varnothing.\ s \vdash e : t}{\Gamma \vdash \lambda x.e : s \to t}$$

$$[\text{Var}]\ \frac{}{\Gamma, x : \forall \vec{\alpha}.\ t \vdash x : t\psi}\ \mathrm{dom}(\psi) \subseteq \vec{\alpha}$$

Similarly to STLC, we can infer the domain of a λ-abstractions and the instantiations of variables using unification.

$$\Gamma \vdash e : t \dashv \psi$$

$$[\text{App}]\ \frac{\Gamma \vdash e_1 : t_1 \dashv \psi_1 \qquad \Gamma\psi_1 \vdash e_2 : t_2 \dashv \psi_2 \qquad \psi = \mathrm{mgu}(t_1\psi_2, t_2 \to \alpha)}{\Gamma \vdash e_1\ e_2 : \alpha\psi \dashv \psi \circ \psi_2 \circ \psi_1}\ \alpha \text{ fresh}$$

$$[\text{Abs}]\ \frac{\Gamma, x : \forall \varnothing.\ \alpha \vdash e : t \dashv \psi}{\Gamma \vdash \lambda x.e : (\alpha\psi) \to t \dashv \psi}\ \alpha \text{ fresh}$$

$$[\text{Var}]\ \frac{}{\Gamma, x : \forall \vec{\alpha}.\ t \vdash x : t\psi \dashv \varnothing}\ \psi \text{ maps type variables in } \vec{\alpha} \text{ to fresh ones}$$

$$[\text{Let}]\ \frac{\Gamma \vdash e_1 : s \dashv \psi_1 \qquad \Gamma\psi_1, x : \forall(\mathrm{fv}(s) \setminus \mathrm{fv}(\Gamma)).\ s \vdash e_2 : t \dashv \psi_2}{\Gamma \vdash \texttt{let}\ x = e_1\ \texttt{in}\ e_2 : t \dashv \psi_2 \circ \psi_1}$$

Exercise: implement a Hindley-Milner type system following these typing rules.

Recap

- **Simply Typed Lambda Calculus** (STLC):
  base types, arrows, products, type variables (but no quantifier)
  $\Rightarrow$ Inference of the domain of functions is possible using unification

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- **Hindley-Milner** (HM): type environments can now quantify universally ($\forall$)
  over some type variables using type-schemes
  $\Rightarrow$ Type inference is almost unchanged
- **System F**: generalization of Hindley-Milner,
  where $\forall$ quantifiers can appear inside type constructors (in particular, arrows)
  $\Rightarrow$ Type inference is not decidable anymore (unless we add some restrictions)

Towards an imperative language

Our λ-calculus is pure, in particular let-variables cannot be reassigned.

Let us extend our language and types with references (i.e. mutable memory cells):

$$
\begin{array}{lrcl}
\textbf{Expressions} & e & ::= & c \mid x \mid \lambda x.e \mid e\,e \mid (e,e) \mid \pi_1 e \mid \pi_2 e \mid e\,\texttt{?}\,e\,\texttt{:}\,e \\
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\end{array}
$$

We can then encode mutable variables with references:

```
mut v = 15 ; v = v + 42 ; ...
```

↓

```
let v = (ref 15) in let _ = (v := !v + 42) in ...
```

$$\text{[Ref]}\ \frac{\Gamma \vdash e : t}{\Gamma \vdash \texttt{ref}\ e : \texttt{ref}\ t} \qquad \text{[Read]}\ \frac{\Gamma \vdash e : \texttt{ref}\ t}{\Gamma \vdash\ !e : t} \qquad \text{[Assign]}\ \frac{\Gamma \vdash x : \texttt{ref}\ t \qquad \Gamma \vdash e : t}{\Gamma \vdash\ x := e : t}$$

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But we have to be careful not to generalize the type of expressions with side-effects:

```
let foo = ref (fun x -> x) in
(* foo: forall 'a. ref('a -> 'a) *)

foo := (fun i -> i + 42) ;
(* Typechecks by instantiating 'a with int *)

!foo false  (* Reduction stuck! false + 42 *)
(* Typechecks by instantiating 'a with false *)
```

$$[\text{Ref}]\ \frac{\Gamma \vdash e : t}{\Gamma \vdash \texttt{ref}\ e : \texttt{ref}\ t} \qquad [\text{Read}]\ \frac{\Gamma \vdash e : \texttt{ref}\ t}{\Gamma \vdash\ !e : t} \qquad [\text{Assign}]\ \frac{\Gamma \vdash x : \texttt{ref}\ t \qquad \Gamma \vdash e : t}{\Gamma \vdash x := e : t}$$

Solution: only generalize values (value restriction).

$$[\text{LetGen}]\ \frac{\Gamma \vdash e_1 : s \qquad \Gamma, x : \forall(\text{fv}(s) \setminus \text{fv}(\Gamma)).\ s \vdash e_2 : t}{\Gamma \vdash \texttt{let}\ x = e_1\ \texttt{in}\ e_2 : t}\ \text{if } e_1 \text{ is a value}$$

$$[\text{Let}]\ \frac{\Gamma \vdash e_1 : s \qquad \Gamma, x : \forall\varnothing.\ s \vdash e_2 : t}{\Gamma \vdash \texttt{let}\ x = e_1\ \texttt{in}\ e_2 : t}\ \text{otherwise}$$

Subtyping and ad-hoc polymorphism

Simply Typed Lambda Calculus (STLC)

Parametric polymorphism (Hindley-Milner)

Subtyping and ad-hoc polymorphism

What is the type of this function?

```
def inv(x:float):
  if x == 0.0:
    return None
  else:
    return 1/x
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What is the type of this one?

```
def f(x:Union[float, NoneType]):
  if x == None:
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  else:
    return x
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What is the type of this function?

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def inv(x:float):
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float→(float∨none)

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def f(x:Union[float, NoneType]):
  if x == None:
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(float∨none)→float

Subtyping

In dynamic languages, functions can manipulate data of heterogeneous types

$\Rightarrow$ Hence we need to be able to express the union of two types $t_1 \vee t_2$

$\Rightarrow$ We may also need to express an intersection, e.g. `printable` $\wedge$ `iterable`

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- The type `none` should be usable everywhere a `float` $\vee$ `none` is expected

Subtyping

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$\Rightarrow$ Hence we need to be able to express the union of two types $t_1 \vee t_2$
$\Rightarrow$ We may also need to express an intersection, e.g. $\texttt{printable} \wedge \texttt{iterable}$

- The type `float` should be usable everywhere a $\texttt{float} \vee \texttt{none}$ is expected
- The type `none` should be usable everywhere a $\texttt{float} \vee \texttt{none}$ is expected

Formally, we define a subtyping relation $\leq$ such that:

- $\leq$ is reflexive ($t \leq t$) and transitive ($t_1 \leq t_2$ and $t_2 \leq t_3$ implies $t_1 \leq t_3$),
- for any t_1 and t_2, $t_1 \leq t_1 \vee t_2$ and $t_2 \leq t_1 \vee t_2$,
- other properties may be desirable, for instance idempotency ($t \simeq t \wedge t$)

Overloaded functions

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def inv(x):
  if isinstance(x, complex):
    return ...
  elif isinstance(x, float):
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Overloaded functions

What is the type of this function?

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def inv(x):
  if isinstance(x, complex):
    return ...
  elif isinstance(x, float):
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```

$$(\texttt{complex} \vee \texttt{float}) \to (\texttt{complex} \vee \texttt{float})$$

or if we want to be more precise

$$(\texttt{complex} \to \texttt{complex}) \wedge (\texttt{float} \to \texttt{float})$$

Parametric polymorphism vs ad-hoc polymorphism

Parametric polymorphism: captures genericity of functions: when the function has the same behavior on (potentially infinitely many) different input types. Implemented by quantifying on type variables (e.g. $\forall \alpha.\ \alpha \to \alpha$).

Parametric polymorphism vs ad-hoc polymorphism

Parametric polymorphism: captures genericity of functions: when the function has the same behavior on (potentially infinitely many) different input types.
Implemented by quantifying on type variables (e.g. $\forall\alpha.\ \alpha \to \alpha$).

Ad-hoc polymorphism: captures overloading of functions: when the function has (finitely many) different behaviors depending on the input type.
Implemented by associating multiple signatures to a top-level function, or using intersection types (e.g. $(\texttt{int} \to \texttt{bool}) \wedge (\texttt{string} \to \texttt{int})$).

Existing approaches

To be continued ...

References i