Polymorphic Type Inference for Dynamic Languages

Reconstructing Types for Systems combining
Parametric, Ad-Hoc, and Subtyping Polymorphism

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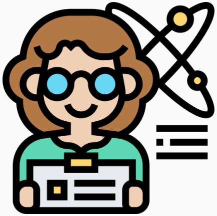
Introduction



country	city	pop	density	⋯
USA	Chicago	2665039	4398	⋯
USA	Boston	675647	2911	⋯
France	Gif-sur-Yvette	22352	1900	⋯
France	Pontamafrey	307	26	⋯
⋮	⋮	⋮	⋮	⋱

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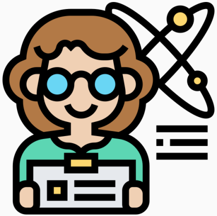

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How to retrieve the population of a city?

get_population in Rust

```rust
fn get_population(data: &str, city: &str) -> Option<u32> {
    let mut rdr = csv::Reader::from_reader(data.as_bytes());
    let city_index = rdr.headers().unwrap().iter()
      .position(|h| h == "city").unwrap();
    let pop_index = rdr.headers().unwrap().iter()
      .position(|h| h == "pop").unwrap();
    for result in rdr.records() {
        let record = result.unwrap();
        if record.get(city_index).unwrap() == v {
            return Some(record.get(pop_index).parse().unwrap());
        }
    }
    None
}
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get_population in Python

```python
def get_population(data, city):
    d = csv.DictReader(StringIO(data))
    for row in d:
        if row['city'] == city:
            return int(row['pop'])
    return None
```

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    return None
```

No need to unwrap or pattern-match the result like in Rust.

In Rust:

```
get_population(data, "Gif-sur-Yvette").unwrap() / 1000          // 22
```

In Python:

```
get_population(data, "Gif-sur-Yvette") // 1000                  # 22
```

get_population in Python

```python
def get_population(data, city):
    d = csv.DictReader(StringIO(data))
    for row in d:
        if row['city'] == city:
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    return None

def in_thousands(n):
    if type(n) is int:
        return n // 1000
    else:
        return None
```

get_population in Python

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def in_thousands(n):
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in_thousands(get_population(data, "Gif-sur-Yvette"))  # 22
in_thousands(get_population(data, "Jpeg-sur-Yvette")) # None
```

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⇒ Unsafe, bad for production code and maintenance of large projects

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My contribution: conception of a type system for a language with type-cases, featuring many forms of polymorphism (parametric, ad-hoc, subtyping) and a type inference.

Summary

Types & Core Language

Declarative Type System

Algorithmic Type System

Reconstruction of the Annotation Tree

Conclusion and Perspective

Types & Core Language

Example: Typing JavaScript's "||" (Logical Or)

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- For `false`, `null`, `0`, `±0.0`, `""`, etc. ⇒ returns `false`
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- For $\underbrace{\mathtt{false}, \mathtt{null}, 0, \pm 0.0, \mathtt{""}, \text{ etc.}}_{\mathtt{Falsy}}$ $\Rightarrow$ returns $\mathtt{false}$
- For $\underbrace{\text{other values}}_{\mathtt{Truthy}}$ $\Rightarrow$ returns $\mathtt{true}$

```
type Falsy = false | null | 0 | 0.0 | ""
type Truthy = ~Falsy
```

$$\mathtt{ToBoolean}: \ (\mathtt{Falsy} \to \mathtt{false}) \wedge (\mathtt{Truthy} \to \mathtt{true})$$

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$$\texttt{LogicalOr}: \quad \forall \alpha, \beta \,. \quad ((\alpha \wedge \texttt{Truthy}, \texttt{Any}) \to \alpha \wedge \texttt{Truthy}) \wedge ((\texttt{Falsy}, \beta) \to \beta)$$

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Challenges:

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- Capture genericity: `LogicalOr` returns its first or second parameter, unchanged

  $\Rightarrow$ parametric polymorphism

Set-Theoretic Types [Frisch, 2004]

Constants	c	$::=$	$\texttt{false} \mid \texttt{true} \mid 0 \mid 1 \mid \ldots$
Basic Types	b	$::=$	$\texttt{Bool} \mid \texttt{Int} \mid \ldots$
Set-Theoretic Types	t	$::=$	$c \mid b \mid t \to t \mid t \times t \mid t \vee t \mid t \wedge t \mid \neg t \mid \texttt{Empty} \mid \texttt{Any}$

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Types are interpreted as sets of values:

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[\![\texttt{false}]\!] = \{\texttt{false}\} & [\![\texttt{Any}]\!] = \mathcal{V} \quad \text{(with } \mathcal{V} \text{ the set of all values)} \\
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Semantic subtyping: $\quad t_1 \leq t_2 \overset{def}{\iff} [\![t_1]\!] \subseteq [\![t_2]\!]$

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Syntax and Semantics

$$\begin{array}{llcl} \textbf{Expressions} & e & ::= & c \,|\, x \,|\, \lambda x.e \,|\, e\,e \,|\, (e,e) \,|\, \pi_i e \,|\, (e \in t)\,?\,e : e \\ \textbf{Values} & v & ::= & c \,|\, \lambda x.e \,|\, (v,v) \end{array}$$

with the usual call-by-value semantics (w/ leftmost outermost strategy):

$$\begin{array}{rcl} (\lambda x.e)v & \rightsquigarrow & e\{v/x\} \\ \pi_1(v_1,v_2) & \rightsquigarrow & v_1 \\ \pi_2(v_1,v_2) & \rightsquigarrow & v_2 \end{array}$$

$$\boxed{\begin{array}{rcll} (v \in t)\,?\,e_1 : e_2 & \rightsquigarrow & e_1 & \text{if } v \text{ has type } t \\ (v \in t)\,?\,e_1 : e_2 & \rightsquigarrow & e_2 & \text{otherwise} \end{array}}$$

Declarative Type System

Usual Rules

$$[\text{Const}]\ \frac{}{\Gamma \vdash c : c} \qquad\qquad [\text{Var}]\ \frac{}{\Gamma \vdash x : \Gamma(x)}\ x \in \text{dom}(\Gamma)$$

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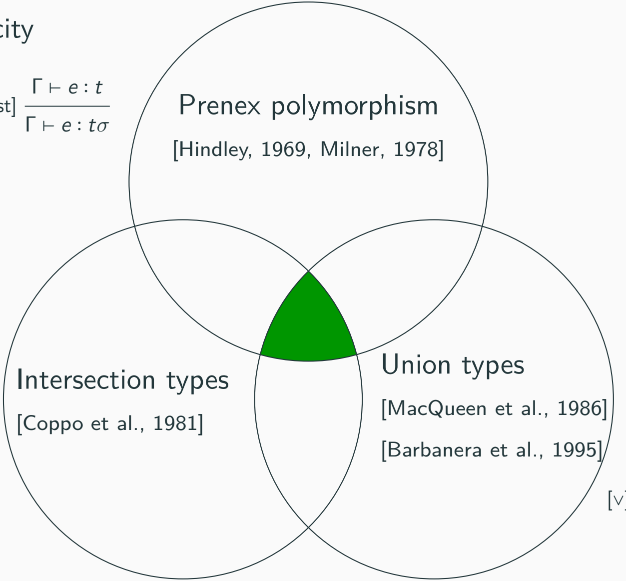
$$[\leq]\ \frac{\Gamma \vdash e : t \qquad t \leq t'}{\Gamma \vdash e : t'}$$

Mixing Union, Intersection, and HM Polymorphism

For genericity

$$[\text{Gen}]\ \frac{\Gamma \vdash e : t}{\Gamma \vdash e : t\{\gamma \leadsto \alpha\}} \qquad [\text{Inst}]\ \frac{\Gamma \vdash e : t}{\Gamma \vdash e : t\sigma}$$



For overloading

$$[\wedge]\ \frac{\Gamma \vdash e : t_1 \qquad \Gamma \vdash e : t_2}{\Gamma \vdash e : t_1 \wedge t_2}$$

For type narrowing

$$[\vee]\ \frac{\Gamma \vdash e' : s_1 \vee s_2 \qquad \Gamma, x : s_1 \vdash e : t \quad \Gamma, x : s_2 \vdash e : t}{\Gamma \vdash e\{e'/x\} : t}$$

Instantiation and Generalization (Hindley Milner)

Some type variables are polymorphic: $\alpha, \beta \in \textbf{Vars}_P$

Some type variables are monomorphic: $\boldsymbol{\gamma}, \boldsymbol{\delta} \in \textbf{Vars}_M$

$$\textbf{Vars} = \textbf{Vars}_P \cup \textbf{Vars}_M$$

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We can instantiate polymorphic type variables:

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We can generalize a monomorphic type variable γ into a polymorphic type variable α (only if γ is not bound to the environment):

$$[\text{Gen}]\ \frac{\Gamma \vdash e : t}{\Gamma \vdash e : t\{\gamma \rightsquigarrow \alpha\}}\ \gamma \notin \text{vars}(\Gamma)$$

$$\begin{aligned}
&\texttt{let id} &&= \lambda x.x \\
&\texttt{let test} &&= (\texttt{id}\ 42, \texttt{id}\ \texttt{true})
\end{aligned}$$

$$
\begin{array}{llll}
\texttt{let} & \texttt{id} & \texttt{=} & \lambda x.x \\
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\end{array}
$$

We first type `id` under the empty environment $\varnothing$:

$$
[\rightarrow\mathrm{I}]\ \dfrac{[\mathrm{Var}]\ \dfrac{}{x : \gamma \vdash x : \gamma}}{\varnothing \vdash \lambda x.x : \gamma \rightarrow \gamma}\ \text{with } \gamma \text{ monomorphic}
$$

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We then type `test` under the environment $\Gamma = (\texttt{id} : \alpha \to \alpha)$:

$$\dfrac{\dfrac{\dfrac{[\mathrm{Var}]}{\Gamma \vdash \texttt{id} : \alpha \to \alpha}}{\Gamma \vdash \texttt{id} : 42 \to 42}\,[\mathrm{Inst}] \quad \dfrac{[\mathrm{Const}]}{\Gamma \vdash 42 : 42}}{\Gamma \vdash \texttt{id}\ 42 : 42}\,[\to\mathrm{E}]$$

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Intersection

Intersection introduction:

$$[\wedge]\ \frac{\Gamma \vdash e : t_1 \qquad \Gamma \vdash e : t_2}{\Gamma \vdash e : t_1 \wedge t_2}$$

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Intersection elimination can be derived from subsumption:

$$[\leq]\ \frac{\Gamma \vdash e : t'}{\Gamma \vdash e : t}\ t' \leq t \qquad \longrightarrow \qquad [\leq]\ \frac{\Gamma \vdash e : t_1 \wedge t_2}{\Gamma \vdash e : t_1}\ t_1 \wedge t_2 \leq t_1$$

For instance, we can type $\lambda x.x$:

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- A first time for the domain `Bool`, yielding `Bool` $\to$ `Bool`,

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$$

For instance, we can type $\lambda x.x$:

- A first time for the domain `Bool`, yielding `Bool` $\to$ `Bool`,
- A second time for the domain `Int`, yielding `Int` $\to$ `Int`,

$$\dfrac{[\mathsf{Var}]\,\dfrac{}{x : \texttt{Bool} \vdash x : \texttt{Bool}}}{\varnothing \vdash \lambda x.x : \texttt{Bool} \to \texttt{Bool}}\,[{\to}\mathsf{I}] \qquad \dfrac{[\mathsf{Var}]\,\dfrac{}{x : \texttt{Int} \vdash x : \texttt{Int}}}{\varnothing \vdash \lambda x.x : \texttt{Int} \to \texttt{Int}}\,[{\to}\mathsf{I}]$$

For instance, we can type $\lambda x.x$:

- A first time for the domain $\texttt{Bool}$, yielding $\texttt{Bool} \to \texttt{Bool}$,
- A second time for the domain $\texttt{Int}$, yielding $\texttt{Int} \to \texttt{Int}$,
- Then, we can use the intersection introduction rule to derive the type $(\texttt{Bool} \to \texttt{Bool}) \wedge (\texttt{Int} \to \texttt{Int})$

$$
\dfrac{
\dfrac{
\dfrac{}{x : \texttt{Bool} \vdash x : \texttt{Bool}}\,[\mathrm{Var}]
}{\varnothing \vdash \lambda x.x : \texttt{Bool} \to \texttt{Bool}}\,[\to\mathrm{I}]
\qquad
\dfrac{
\dfrac{}{x : \texttt{Int} \vdash x : \texttt{Int}}\,[\mathrm{Var}]
}{\varnothing \vdash \lambda x.x : \texttt{Int} \to \texttt{Int}}\,[\to\mathrm{I}]
}{\varnothing \vdash \lambda x.x : (\texttt{Bool} \to \texttt{Bool}) \wedge (\texttt{Int} \to \texttt{Int})}\,[\wedge]
$$

Union

Union introduction can be derived from subsumption:

$$[\leq]\ \frac{\Gamma \vdash e : t'}{\Gamma \vdash e : t}\ t' \leq t \quad \longrightarrow \quad [\leq]\ \frac{\Gamma \vdash e : t_1}{\Gamma \vdash e : t_1 \vee t_2}\ t_1 \leq t_1 \vee t_2$$

Union

Union introduction can be derived from subsumption:

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Union elimination:

$$[\vee]\ \frac{\Gamma \vdash e' : s_1 \vee s_2 \qquad \Gamma, x : s_1 \vdash e : t \quad \Gamma, x : s_2 \vdash e : t}{\Gamma \vdash e\{e'/x\} : t}$$

$$(\text{ f } 42 \text{ , } \text{ f } 42 \text{ }) \qquad \text{with } f : \texttt{Int} \to \texttt{Bool}$$

$$(\underbrace{\mathsf{f}\ 42}_{x}, \underbrace{\mathsf{f}\ 42}_{x}) \qquad \text{with } f : \mathtt{Int} \to \mathtt{Bool}$$

$$\text{with } x : \mathtt{Bool} \simeq \mathtt{true} \vee \mathtt{false}$$

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We type (x, x):

- First, by assuming that $x : \mathtt{true} \quad \Rightarrow \quad \mathtt{true} \times \mathtt{true}$,

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- Then, by assuming that $x : \mathtt{false} \quad \Rightarrow \quad \mathtt{false} \times \mathtt{false}$

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$$[\vee]\ \frac{\Gamma \vdash f\ 42 : \texttt{true} \vee \texttt{false} \qquad \Gamma, x : \texttt{true} \vdash (x, x) : \texttt{true} \times \texttt{true} \qquad \Gamma, x : \texttt{false} \vdash (x, x) : \texttt{false} \times \texttt{false}}{\Gamma \vdash (x, x)\{(f\ 42)/x\} : (\texttt{true} \times \texttt{true}) \vee (\texttt{false} \times \texttt{false})}$$

Unsound in the presence of polymorphic type variables:

$$(\ \mathsf{f}\ 42\ ,\ \mathsf{f}\ 42\ ) \qquad \text{with } f : \mathtt{Int} \to \mathtt{Bool}$$

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- Then, by assuming that $x : \mathtt{Bool} \wedge \neg\alpha \quad \Rightarrow \quad \mathtt{Empty}$ (by substituting α by $\mathtt{Any}$)

We must prevent the type decomposition from containing polymorphic type variables:

$$[\vee]\ \frac{\Gamma \vdash e' : s \qquad \Gamma, x : s \wedge u \vdash e : t \quad \Gamma, x : s \wedge \neg u \vdash e : t}{\Gamma \vdash e\{e'/x\} : t}$$

where u does not contain any polymorphic type variable: $\mathrm{vars}(u) \cap \mathbf{Vars}_P = \varnothing$

Typing Type-Cases

Two cases:

$$\boxed{\begin{array}{llll} (v{\in}t)\ ?\ e_1 : e_2 & \rightsquigarrow & e_1 & \text{if } v \text{ has type } t \\ (v{\in}t)\ ?\ e_1 : e_2 & \rightsquigarrow & e_2 & \text{otherwise} \end{array}}$$

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Two rules:

$$[\epsilon_1]\ \frac{\Gamma \vdash e : t \qquad \Gamma \vdash e_1 : t_1}{\Gamma \vdash (e{\in}t)\ ?\ e_1 : e_2 : t_1} \qquad [\epsilon_2]\ \frac{\Gamma \vdash e : \neg t \qquad \Gamma \vdash e_2 : t_2}{\Gamma \vdash (e{\in}t)\ ?\ e_1 : e_2 : t_2}$$

Typing Type-Cases: Union Elimination and Type Narrowing

$$[\vee]\ \frac{\begin{array}{c}\Gamma \vdash e' : s \\ \Gamma, x : s \wedge u \vdash e : t \quad \Gamma, x : s \wedge \neg u \vdash e : t\end{array}}{\Gamma \vdash e\{e'/x\} : t} \qquad [\in_1]\ \frac{\Gamma \vdash e : t \qquad \Gamma \vdash e_1 : t_1}{\Gamma \vdash (e \in t)\ ?\ e_1 : e_2 : t_1} \qquad [\in_2]\ \frac{\Gamma \vdash e : \neg t \qquad \Gamma \vdash e_2 : t_2}{\Gamma \vdash (e \in t)\ ?\ e_1 : e_2 : t_2}$$

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$$\lambda x.\ (\ \mathtt{x} \in \mathtt{Int})\,?\ \ \mathtt{x}\ + 1 : \mathtt{false}$$

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$$\Gamma = \{\ x : \texttt{Any}\ \} \qquad \lambda x.\ (\ \texttt{x} \in \texttt{Int})\ ?\ \ \texttt{x} + 1 : \texttt{false}$$

$$[\to\text{I}]\ \frac{x : \texttt{Any} \vdash (x \in \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Int} \vee \texttt{false}}{\varnothing \vdash \lambda x.\ (x \in \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Any} \to (\texttt{Int} \vee \texttt{false})}$$

Typing Type-Cases: Union Elimination and Type Narrowing

$$[\vee]\ \frac{\Gamma \vdash e' : s \qquad \Gamma, x : s \wedge u \vdash e : t \quad \Gamma, x : s \wedge \neg u \vdash e : t}{\Gamma \vdash e\{e'/x\} : t}$$

$$[\epsilon_1]\ \frac{\Gamma \vdash e : t \qquad \Gamma \vdash e_1 : t_1}{\Gamma \vdash (e \epsilon t)\ ?\ e_1 : e_2 : t_1} \qquad [\epsilon_2]\ \frac{\Gamma \vdash e : \neg t \qquad \Gamma \vdash e_2 : t_2}{\Gamma \vdash (e \epsilon t)\ ?\ e_1 : e_2 : t_2}$$

$$\Gamma = \{\ x : \underbrace{\texttt{Any}}_{\texttt{Int} \vee \neg\texttt{Int}}\ \} \qquad \lambda x.\ (\texttt{ x } \in \texttt{Int})\ ?\ \texttt{ x } + 1 : \texttt{false}$$

$$[\to\mathrm{I}]\ \dfrac{[\vee]\ \dfrac{x : \texttt{Int} \vdash (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Int} \qquad x : \neg\texttt{Int} \vdash (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{false}}{x : \texttt{Any} \vdash (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Int} \vee \texttt{false}}}{\varnothing \vdash \lambda x.\ (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Any} \to (\texttt{Int} \vee \texttt{false})}$$

Typing Type-Cases: Union Elimination and Type Narrowing

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$$[\to\mathrm{I}]\ \dfrac{[\vee]\ \dfrac{[\epsilon_1]\ \dfrac{x : \texttt{Int} \vdash x + 1 : \texttt{Int}}{x : \texttt{Int} \vdash (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Int}} \qquad x : \neg\texttt{Int} \vdash (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{false}}{x : \texttt{Any} \vdash (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Int} \vee \texttt{false}}}{\varnothing \vdash \lambda x.\ (x \epsilon \texttt{Int})\ ?\ x + 1 : \texttt{false} : \texttt{Any} \to (\texttt{Int} \vee \texttt{false})}$$

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$$\Gamma \;=\; \{\ x : \underbrace{\texttt{Any}}_{\texttt{Int} \vee \neg\texttt{Int}}\ \}$$

$$\lambda x.\ (\underbrace{x}_{\neg\texttt{Int}} \in \texttt{Int})\ ?\ x+1 : \texttt{false}$$

$$[\to\text{I}]\ \dfrac{[\vee]\ \dfrac{[\epsilon_1]\ \dfrac{x : \texttt{Int} \vdash x+1 : \texttt{Int}}{x : \texttt{Int} \vdash (x\epsilon\texttt{Int})\ ?\ x+1 : \texttt{false} : \texttt{Int}} \qquad [\epsilon_2]\ \dfrac{x : \neg\texttt{Int} \vdash \texttt{false} : \texttt{false}}{x : \neg\texttt{Int} \vdash (x\epsilon\texttt{Int})\ ?\ x+1 : \texttt{false} : \texttt{false}}}{x : \texttt{Any} \vdash (x\epsilon\texttt{Int})\ ?\ x+1 : \texttt{false} : \texttt{Int} \vee \texttt{false}}}{\varnothing \vdash \lambda x.\ (x\epsilon\texttt{Int})\ ?\ x+1 : \texttt{false} : \texttt{Any} \to (\texttt{Int} \vee \texttt{false})}$$

Typing Type-Cases: Union Elimination and Type Narrowing

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$$[\to\mathrm{I}]\ \dfrac{[\vee]\ \dfrac{[\epsilon_1]\ \dfrac{x : \mathtt{Int} \vdash x + 1 : \mathtt{Int}}{x : \mathtt{Int} \vdash (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \mathtt{Int}} \qquad [\epsilon_2]\ \dfrac{x : \neg\mathtt{Int} \vdash \mathtt{false} : \mathtt{false}}{x : \neg\mathtt{Int} \vdash (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \mathtt{false}}}{x : \mathtt{Any} \vdash (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \mathtt{Int} \vee \mathtt{false}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \mathtt{Any} \to (\mathtt{Int} \vee \mathtt{false})}$$

Capturing Overloaded Behaviors: Intersection Introduction

$$[\wedge]\ \frac{\Gamma \vdash e : t_1 \quad \Gamma \vdash e : t_2}{\Gamma \vdash e : t_1 \wedge t_2} \qquad [\in_1]\ \frac{\Gamma \vdash e : t \qquad \Gamma \vdash e_1 : t_1}{\Gamma \vdash (e \in t)\ ?\ e_1 : e_2 : t_1} \qquad [\in_2]\ \frac{\Gamma \vdash e : \neg t \qquad \Gamma \vdash e_2 : t_2}{\Gamma \vdash (e \in t)\ ?\ e_1 : e_2 : t_2}$$

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$$\lambda x.\ (x \epsilon \mathtt{Int})\ ?\ x + 1 : \mathtt{false}$$

Capturing Overloaded Behaviors: Intersection Introduction

$$[\wedge]\ \frac{\Gamma \vdash e : t_1 \quad \Gamma \vdash e : t_2}{\Gamma \vdash e : t_1 \wedge t_2} \qquad [\epsilon_1]\ \frac{\Gamma \vdash e : t \qquad \Gamma \vdash e_1 : t_1}{\Gamma \vdash (e \epsilon t)\ ?\ e_1 \mathbin{:} e_2 : t_1} \qquad [\epsilon_2]\ \frac{\Gamma \vdash e : \neg t \qquad \Gamma \vdash e_2 : t_2}{\Gamma \vdash (e \epsilon t)\ ?\ e_1 \mathbin{:} e_2 : t_2}$$

$$\Gamma \ = \ \{\ x : \texttt{Int}\ \} \qquad\qquad \lambda x.\ (x \epsilon \texttt{Int})\ ?\ x + 1 \ \mathbf{:}\ \texttt{false}$$

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$$\lambda x.\ (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false}$$

$$[\wedge]\ \dfrac{[\to\mathrm{I}]\ \dfrac{[\epsilon_1]\ \dfrac{x : \mathtt{Int} \vdash x + 1 : \mathtt{Int}}{x : \mathtt{Int} \vdash (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \mathtt{Int}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \mathtt{Int} \to \mathtt{Int}} \qquad [\to\mathrm{I}]\ \dfrac{[\epsilon_2]\ \dfrac{x : \neg\mathtt{Int} \vdash \mathtt{false} : \mathtt{false}}{x : \neg\mathtt{Int} \vdash (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \mathtt{false}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : \neg\mathtt{Int} \to \mathtt{false}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\ ?\ x + 1 : \mathtt{false} : (\mathtt{Int} \to \mathtt{Int}) \wedge (\neg\mathtt{Int} \to \mathtt{false})}$$

$$(\mathtt{Int} \to \mathtt{Int}) \wedge (\neg\mathtt{Int} \to \mathtt{false}) \quad \leq \quad \mathtt{Any} \to (\mathtt{Int} \vee \mathtt{false})$$

Contribution 1

Type safety of the declarative type system

For every expression e, if $\varnothing \vdash e : t$, then:

– either e reduces to a value v of type t,

– or e diverges.

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However, this type system is not algorithmic.

How to turn it into an algorithm?

Algorithmic Type System

Declarative = Non-algorithmic

$$[\wedge]\dfrac{[\to\mathrm{I}]\dfrac{[\epsilon_1]\dfrac{x : \mathtt{Int} \vdash x+1 : \mathtt{Int}}{x : \mathtt{Int} \vdash (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \mathtt{Int}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \mathtt{Int} \to \mathtt{Int}} \qquad [\to\mathrm{I}]\dfrac{[\epsilon_2]\dfrac{x : \neg\mathtt{Int} \vdash \mathtt{false} : \mathtt{false}}{x : \neg\mathtt{Int} \vdash (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \mathtt{false}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \neg\mathtt{Int} \to \mathtt{false}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : (\mathtt{Int} \to \mathtt{Int}) \wedge (\neg\mathtt{Int} \to \mathtt{false})}$$

$$[\to\mathrm{I}]\dfrac{[\vee]\dfrac{[\epsilon_1]\dfrac{x : \mathtt{Int} \vdash x+1 : \mathtt{Int}}{x : \mathtt{Int} \vdash (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \mathtt{Int}} \qquad [\epsilon_2]\dfrac{x : \neg\mathtt{Int} \vdash \mathtt{false} : \mathtt{false}}{x : \neg\mathtt{Int} \vdash (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \mathtt{false}}}{x : \mathtt{Any} \vdash (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \mathtt{Int} \vee \mathtt{false}}}{\varnothing \vdash \lambda x.\ (x\epsilon\mathtt{Int})\,?\,x+1 : \mathtt{false} : \mathtt{Any} \to (\mathtt{Int} \vee \mathtt{false})}$$

Many possible derivations:

Declarative = Non-algorithmic

$$\dfrac{\dfrac{\dfrac{x : \texttt{Int} \vdash x+1 : \texttt{Int}}{x : \texttt{Int} \vdash (x\epsilon\texttt{Int})\,?\,x+1 : \texttt{false} : \texttt{Int}}[\epsilon_1]}{\varnothing \vdash \lambda x.\ (x\epsilon\texttt{Int})\,?\,x+1 : \texttt{false} : \texttt{Int} \to \texttt{Int}}[\to\mathrm{I}] \quad \dfrac{\dfrac{x : \neg\texttt{Int} \vdash \texttt{false} : \texttt{false}}{x : \neg\texttt{Int} \vdash (x\epsilon\texttt{Int})\,?\,x+1 : \texttt{false} : \texttt{false}}[\epsilon_2]}{\varnothing \vdash \lambda x.\ (x\epsilon\texttt{Int})\,?\,x+1 : \texttt{false} : \neg\texttt{Int} \to \texttt{false}}[\to\mathrm{I}]}{\varnothing \vdash \lambda x.\ (x\epsilon\texttt{Int})\,?\,x+1 : \texttt{false} : (\texttt{Int} \to \texttt{Int}) \wedge (\neg\texttt{Int} \to \texttt{false})}[\wedge]$$

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Many possible derivations:

- Some rules can be applied on every expression (the system is not syntax-directed):
  - Union elimination [$\vee$]
  - Intersection introduction [$\wedge$]
  - Instantiation [Inst]
  - Subsumption [$\leq$]

Declarative = Non-algorithmic

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How to make the type system algorithmic?

Making the Type System Syntax-Directed

Solution to make the type system syntax directed without loosing generality:

- Subsumption [$\leq$] and instantiation [Inst] are embedded in destructor rules:

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$$[\to\text{E}]\ \frac{\Gamma \vdash e_1 : s \to t \qquad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1 e_2 : t} \quad + \quad [\text{Inst}]\ \frac{\Gamma \vdash e : t}{\Gamma \vdash e : t\sigma} \quad + \quad [\leq]\ \frac{\Gamma \vdash e : t \qquad t \leq t'}{\Gamma \vdash e : t'}$$

$$\Rightarrow \quad [\text{App}]\ \frac{\Gamma \vdash e_1 : t_1 \text{ with } t_1\sigma_1 \leq s \to t \qquad \Gamma \vdash e_2 : t_2 \text{ with } t_2\sigma_2 \leq s}{\Gamma \vdash e_1 e_2 : t}$$

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$$[\vee]\ \frac{\Gamma \vdash e' : s \qquad \Gamma, x : s \wedge u \vdash e : t \quad \Gamma, x : s \wedge \neg u \vdash e : t}{\Gamma \vdash e\{e'/x\} : t}$$

$$\Rightarrow \quad [\text{Bind}]\ \frac{\Gamma \vdash a : s \qquad (\forall i \in I)\ \Gamma, \mathbf{u} : s \wedge u_i \vdash \kappa : t_i}{\Gamma \vdash \ \texttt{bind}\ \mathbf{u}\ \texttt{=}\ a\ \texttt{in}\ \kappa\ : \bigvee_{i \in I} t_i}\ \{u_i\}_{i \in I}\ \text{a partition of}\ \texttt{Any}$$

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$$[\ \underbrace{\texttt{bind}\ \mathbf{u} = a\ \texttt{in}\ \kappa}_{\text{MSC}}\ \ |\ \ \underbrace{[\vee]\ (\ldots,\{(\texttt{Int},\ldots),(\neg\texttt{Int},\ldots)\})}_{\text{annotation tree}}\ ]$$

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$$\downarrow$$

$$[\text{Bind-Alg}]\ \dfrac{\dfrac{\cdots}{\Gamma \vdash a : s} \quad \dfrac{\cdots}{\Gamma, \mathbf{u} : s \wedge \texttt{Int} \vdash e : t_1} \quad \dfrac{\cdots}{\Gamma, \mathbf{u} : s \wedge \neg\texttt{Int} \vdash e : t_2}}{\Gamma \vdash [\texttt{bind}\ \mathbf{u}\ \texttt{=}\ a\ \texttt{in}\ \kappa \mid [\vee]\ (\ldots,\{(\texttt{Int},\ldots),(\neg\texttt{Int},\ldots)\})] : t_1 \vee t_2}$$

Contribution 2

Equivalence between declarative and algorithmic type system

e is typeable with the declarative type system

if and only if

there exists an annotation such that $\mathsf{MSC}(e)$ is typeable with the algorithmic system.

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But how to infer annotation trees?

Reconstruction of the Annotation Tree

Reconstruction of Type Decompositions

- We use type-cases to deduce how to decompose union types:
  when encountering `(z∈Int)?x:y`,
  we backtrack to the `bind` definition of `z` and split its type into `Int` ; `¬Int`

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$$(\mathtt{id}\ x{\in}\mathtt{Int})\,?\,x:\mathtt{false} \quad \text{with} \quad \mathtt{id}:\alpha\to\alpha \ \ \text{and}\ \ x:\mathtt{Any}$$

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```
bind x = x in
bind y = false in
bind z = id x in
bind u = (z∈Int) ? x : y in
u
```

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  when encountering $(\mathbf{z}\in\texttt{Int})\,?\,\mathbf{x}:\mathbf{y}$,
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$\texttt{bind}\,\mathbf{x} : \texttt{Any} = x\ \texttt{in}$

$\texttt{bind}\,\mathbf{y} : \texttt{false} = \texttt{false}\ \texttt{in}$

$\texttt{bind}\,\mathbf{z} : \texttt{Any} = \texttt{id}\ \mathbf{x}\ \texttt{in}$

$\texttt{bind}\,\mathbf{u} = (\mathbf{z}\in\texttt{Int})\,?\,\mathbf{x}:\mathbf{y}\ \texttt{in}$

$\mathbf{u}$

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```
bind x : Any = x in
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$\texttt{bind}\,\mathbf{x} : \texttt{Any} = x\ \texttt{in}$

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$\texttt{bind}\,\mathbf{z} : \texttt{Int} \,;\, \neg\texttt{Int} = \texttt{id}\ \mathbf{x}\ \texttt{in}$

$\texttt{bind}\,\mathbf{u} = (\mathbf{z}{\in}\texttt{Int})\,?\,\mathbf{x}:\mathbf{y}\ \texttt{in}$

$\mathbf{u}$

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```
bind x : Int ; ¬Int = x in
bind y : false = false in
bind z : Int ; ¬Int = id x in
bind u = (z∈Int) ? x : y in
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```

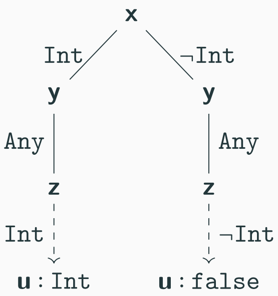
Reconstruction of Type Decompositions

- We use type-cases to deduce how to decompose union types:
  when encountering (**z**∈Int) ? **x** : **y**,
  we backtrack to the `bind` definition of **z** and split its type into Int ; ¬Int

- Then, we backpropagate this split on the variables used in the definition of **z**.

$$(\mathtt{id}\ x \in \mathtt{Int})\ ?\ x : \mathtt{false} \quad \text{with} \quad \mathtt{id} : \alpha \to \alpha \ \text{ and } \ x : \mathtt{Any}$$

```
bind x : Int ; ¬Int = x in
bind y : false = false in
bind z : Int ; ¬Int = id x in
bind u = (z∈Int) ? x : y in
u
```



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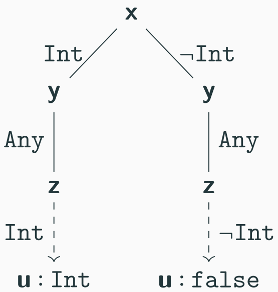

$\mathbf{u} : \texttt{Int} \vee \texttt{false} \ \Leftarrow$

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- Each solution is considered in a separate branch.

Reconstruction of the Type of Parameters (example)

```
function LogicalOr (x:γ, y:δ) {
  if (ToBoolean(x)) { return x; } else { return y; }
}
```

with $\mathtt{ToBoolean}: (\mathtt{Truthy} \rightarrow \mathtt{true}) \wedge (\mathtt{Falsy} \rightarrow \mathtt{false})$

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$\{\gamma \rightsquigarrow \gamma' \wedge \texttt{Truthy} \ ; \ \alpha \rightsquigarrow \alpha' \vee \texttt{true}\} \ ; \ \{\gamma \rightsquigarrow \gamma'' \wedge \texttt{Falsy} \ ; \ \alpha \rightsquigarrow \alpha'' \vee \texttt{false}\}$

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function LogicalOr (x: {γ′ ∧ Truthy ; γ″ ∧ Falsy}, y:δ) {
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$$(\gamma' \wedge \texttt{Truthy}, \delta)$$

$$\Downarrow$$

$$\gamma' \wedge \texttt{Truthy}$$

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$\Downarrow$	$\Downarrow$
$\gamma' \wedge \texttt{Truthy}$	δ

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$$\textbf{function}\ \texttt{LogicalOr (x: \{}\gamma' \wedge \texttt{Truthy ; }\gamma'' \wedge \texttt{Falsy\}, y:}\delta\texttt{) \{}$$
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$\Downarrow$	$\Downarrow$
$\gamma' \wedge \texttt{Truthy}$	δ

$$((\gamma' \wedge \texttt{Truthy}, \delta) \to \gamma' \wedge \texttt{Truthy}) \ \wedge \ ((\gamma'' \wedge \texttt{Falsy}, \delta) \to \delta)$$

Contribution 3

- Conception of a sound and terminating (but incomplete) algorithm to reconstruct annotation trees, using tallying and backtracking

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- Several extensions: pattern matching, records, regular expression types (lists)
- Several optimizations: tree pruning, memoization, type simplification

Demo (https://www.cduce.org/dynlang/)

```
type Falsy = False | "" | 0 | Null
type Truthy = ~Falsy

let to_boolean x =
  if x is Truthy then true else false
```
type> $(\mathtt{Truthy} \to \mathtt{true}) \;\; \wedge \;\; (\mathtt{Falsy} \to \mathtt{false})$

```
let logical_or (x,y) = if to_boolean x then x else y
```
type> $((\alpha \wedge \mathtt{Truthy}, \mathtt{Any}) \to \alpha \wedge \mathtt{Truthy}) \;\; \wedge \;\; ((\mathtt{Falsy}, \beta) \to \beta)$

```
let id x = logical_or (x,x)
```
type> $\alpha \to \alpha$

Demo (https://www.cduce.org/dynlang/)

```
let fixpoint = fun f ->
  let delta = fun x -> f ( fun  v -> x x v ) in
  delta delta
```

type> $((\alpha \to \beta) \to (\alpha \to \beta) \wedge \gamma) \to (\alpha \to \beta) \wedge \gamma$

```
let map_stub map f lst =
  match lst with
  | [] -> []
  | (e,lst) -> (f e, map f lst)
```

```
let map = fixpoint map_stub
```

type> $(\mathtt{Any} \to [\,] \to [\,]) \;\wedge\; ((\alpha \to \beta) \to [\alpha+] \to [\beta+])$

Demo (https://www.cduce.org/dynlang/)

```
let rec filter (f: (α → Any) ∧ (β → Falsy)) (l: [(α ∨ β)*]) =
  match l with
  | [] -> []
  | (e,l) -> if f e is Truthy then (e, filter f l) else filter f l
  end
type> (α → Any) ∧ (β → Falsy) → [(α ∨ β)*] → [(α ∖ β)*]

let filtered_list = filter to_boolean [42;37;null;42;"";4]
type> [(4 ∨ 37 ∨ 42)*]

let test = map ((+)1) filtered_list
type> [Int*]
```

Conclusion and Perspective

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Publications:

- Science of Computer Programming: "Revisiting occurrence typing"
- POPL'22: "On Type-Cases, Union Elimination, and Occurrence Typing"
- POPL'24: "Polymorphic Type Inference for Dynamic Languages"

Future Work

Which features do we support?

- Overloaded functions, dynamic dispatch (type-cases)
- Generics (parametric polymorphism)
- Structural subtyping (pairs, records)

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Which features are missing?

- Nominal subtyping (abstract data types)
- Mutability of the state (references)
- Gradual typing, for a seamless integration and even more flexibility
- Language-specific features
  (example: pattern guards in `Elixir` [Castagna et al., 2023])